Results for some inversion problems for classical continuous and discrete orthogonal polynomials

1997 J. Phys. A: Math. Gen. 30 L35

(http://iopscience.iop.org/0305-4470/30/3/002)

J. Phys. A: Math. Gen. **30** (1997) L35–L40. Printed in the UK PII: S0305-4470(97)77788-6

LETTER TO THE EDITOR

Results for some inversion problems for classical continuous and discrete orthogonal polynomials

A Zarzo†‖, I Area‡¶, E Godoy‡+ and A Ronveaux§*

† Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, c/ José Gutiérrez Abascal 2, 28006 Madrid, Spain
‡ Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales y Minas, Universidad de Vigo, 36200-Vigo, Spain
§ Mathematical Physics, Facultés Universitaires Notre-Dame de la Paix, B-5000 Namur, Belgium

Received 10 September 1996

Abstract. Explicit expressions for the coefficients in the expansion of classical discrete orthogonal polynomials (Charlier, Meixner, Krawtchouck, Hahn, Hahn–Eberlein) into the falling factorial basis are given. The corresponding inversion problems are solved explicitly. This is done by using a general algorithm, recently developed by the authors, which is also applied to this kind of inversion problem but relating the x^n basis and the classical (continuous) orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel.

Given a polynomial family $\{P_n(x)\}_{n=0}^{M}$ defined by the coefficients $a_{n,m}$ in the expansion

$$P_n(x) = \sum_{m=0}^{n} a_{n,m}\, x^m \qquad (n = 0, 1, \ldots, M)$$

the corresponding inversion problem requires inversion of the triangular $(M+1)\times(M+1)$ matrix $(a_{n,m})$ in order to represent x^n in the $P_m(x)$ basis:

$$x^n = \sum_{m=0}^{n} \widetilde{a}_{n,m}\, P_m(x). \tag{1}$$

When the P_n-family belongs to the so-called classical orthogonal polynomials (continuous or discrete) some inverse formulae can be obtained [7] by using the representation of $P_n(x)$ in terms of hypergeometric functions.

Recently [8] (see also [12]), in the case of classical *discrete* orthogonal polynomials (Charlier, Meixner, Krawtchouck, Hahn and Hahn–Eberlein) the falling factorial basis, $x^{[0]} = 1$ and $x^{[k]} = (-1)^k(-x)_k = x(x-1)(x-2)\cdots(x-k+1)$ ($(x)_k$ being the well known Pochhammer symbol), has been suggested to be more natural than the x^n basis, mainly when these polynomials appear in several problems related to combinatorics and graph theory [9] and also in quantum mechanics [8, 12].

‖ E-mail: azarzo@ccupm.upm.es
¶ E-mail: area@dma.uvigo.es
+ E-mail: egodoy@dma.uvigo.es
* E-mail: andre.ronveaux@fundp.ac.be

In [8, p 53] (although explicit solutions are not given) the authors point out that the *direct problem*, i.e. the computation of the coefficients $D_m(n)$ in the expansion

$$P_n(x) = \sum_{m=0}^{n} D_m(n) x^{[m]} \tag{2}$$

could be solved by using the expression of the classical discrete orthogonal polynomials in terms of hypergeometric functions. However, there is no information concerning the solution of the corresponding *inverse problem*, i.e. obtaining the coefficients $I_m(n)$ in the expansion

$$x^{[n]} = \sum_{m=0}^{n} I_m(n) P_m(x). \tag{3}$$

The computation of both the coefficients $D_m(n)$ in (2) and $I_m(n)$ in (3) are particular cases of the so-called general connection problem between two families of polynomials $\{P_n(x)\}$ and $\{Q_m(x)\}$, which consists of computing the coefficients $C_m(n)$ in the expansion

$$P_n(x) = \sum_{m=0}^{n} C_m(n)\ Q_m(x). \tag{4}$$

Under certain conditions [3, 5, 10, 11], this general problem can be solved in a recurrent way by using an algorithm developed recently by the authors (see, e.g., [11]), which can be summarized as follows. Assume that:
(a) The polynomial $P_n(x)$ in (4) satisfies a difference equation

$$\mathcal{L}_r[P_n(x)] := \sum_{i=0}^{r} A_i(x; n) \Delta^i P_n(x) = 0 \qquad (\Delta f(x) = f(x+1) - f(x))$$

where the coefficients $A_i(x; n)$ are polynomials in x of fixed degree (i.e. the degree does not depend on n).
(b) The family $\{Q_m(x)\}$ in (4) satisfies a finite $(h+2)$-term recurrence relation

$$x Q_m(x) = \sum_{k=m-h}^{m+1} B_{m,k} Q_k(x) \tag{5}$$

where the $B_{m,k}$ are coefficients that are independent of x. Moreover, this Q_m-family also satisfies a finite structure relation

$$p(x) \Delta Q_m(x) = \sum_{k=m-s-1}^{m+t-1} F_{m,k} Q_k(x) \tag{6}$$

where the $F_{m,k}$ are constants, s is a fixed integer and $p(x)$ is a polynomial of degree t.

Then, the action of the rth-order difference operator $\mathcal{L}_r$ on both sides of the connection problem (4), gives rise to the relation

$$\sum_{m=0}^{n} C_m(n) \mathcal{L}_r[Q_m(x)] = 0 \tag{7}$$

which contains terms of the form $x^j \Delta^i Q_m(x)$, where i runs from 0 to r and j depends upon the degree of the polynomial coefficients characterizing the operator $\mathcal{L}_r$. The appropriate (and possibly repeated) use of the properties (5), (6) allows us to express all terms appearing in the latter sum as a linear combination (with constant coefficients) of the polynomials $Q_k(x)$ themselves. Thus, if the family $\{Q_m(x)\}$ satisfies equations (5), (6), it is always possible to transform (7) into a relation of the form $\sum_{m=0}^{K} C_m(n) R_m[Q_m(x)] = 0$. Here, K

is a positive integer whose specific value depends on the operator $\mathcal{L}_r$ and also on relations (5), (6) (see [11]), and R_m denotes a linear operator with constant (independent of x) coefficients acting on the index m. Then, a simple shift of indices leads to the expression

$$\sum_{m=0}^{K} M_m[C_m(n)]Q_m(x) = 0 \quad \text{then} \quad M_m[C_m(n)] = 0 \qquad (m = 0, \ldots, K).$$

Here M_m denotes a linear operator with constant coefficients acting on m. Thus in this way a linear system of equations satisfied by the connection coefficients is obtained. Finally, due to its particular structure (see [11]), from this linear system a recurrence relation (in the index m only) can easily be devised for the coefficients $C_m(n)$.

Let us consider the *direct problem* (2). It is well known [8] that the classical discrete orthogonal polynomials are solutions of a second-order difference equation given by

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0 \qquad (\nabla f(x) = f(x) - f(x-1))$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of degree at most 2 and 1, respectively, and λ_n is a constant. Moreover, the falling factorial basis satisfies the following relations of the type (5), (6):

$$\begin{aligned} x\, x^{[m-1]} &= x^{[m]} + (m-1)x^{[m-1]} && (m \geqslant 1) \\ \Delta x^{[m]} &= m x^{[m-1]} && (m \geqslant 1) \\ \nabla x^{[m]} &= m(x-1)^{[m-1]} && (m \geqslant 1). \end{aligned}$$

Clearly, requirements (a) and (b) are fulfilled in this case, and so our algorithm provides a recurrence relation for the coefficients $D_m(n)$. Using monic polynomials and the basic data given in [8], these two term recurrences are solved. The corresponding coefficients $D_m(n)$ for each monic classical discrete family are listed in table 1.

Table 1. Solutions of the direct problem (2) for all monic classical discrete orthogonal polynomials.

$P_n(x)$	$D_m(n) \qquad (0 \leqslant m \leqslant n)$
Charlier: $c_n^{(\mu)}(x)$	$\binom{n}{m}(-\mu)^{n-m}$
Meixner: $m_n^{(\gamma,\mu)}(x)$	$\binom{n}{m}\left(\frac{\mu}{\mu-1}\right)^{n-m}(n+\gamma-1)^{[n-m]}$
Krawtchouck: $k_n^{(p)}(x; N)$	$\binom{n}{m}(-p)^{n-m}(N-n+1)_{n-m}$
Hahn: $h_n^{(\alpha,\beta)}(x; N)$	$\binom{n}{m}\frac{(n-N)^{[n-m]}(n+\beta)^{[n-m]}}{(2n+\alpha+\beta)^{[n-m]}}$
Hahn–Eberlein: $\tilde{h}_n^{(\mu,\nu)}(x; N)$	$(-1)^{n-m}\binom{n}{m}\frac{(N-n)^{[n-m]}(N+\nu-n)_{n-m}}{(\mu+\nu+2N-2n)_{n-m}}$

In the *inverse problem* (3), the falling factorial basis verifies the following first-order difference equation:

$$\mathcal{L}_1[x^{[n]}] \equiv (x-n+1)\ \Delta(x^{[n]}) - n\ x^{[n]} = 0.$$

Moreover all classical discrete orthogonal polynomials satisfy a three-term recurrence relation (i.e. a relation of type (5) with $h = 1$) and also a structure relation (i.e. a relation of type (6) with $s = 0$ and $t \leqslant 2$). Then, our algorithm can also be used in this case.

Table 2. Solutions of the inverse problem (3) for all monic classical discrete orthogonal polynomials.

$P_m(x)$	$I_m(n) \qquad (0 \leqslant m \leqslant n)$
Charlier: $c_m^{(\mu)}(x)$	$\binom{n}{m}\mu^{n-m}$
Meixner: $m_m^{(\gamma,\mu)}(x)$	$\binom{n}{m}\left(\frac{\mu}{1-\mu}\right)^{n-m}(\gamma+m)_{n-m}$
Krawtchouck: $k_m^{(p)}(x;N)$	$\binom{n}{m}p^{n-m}(N-n+1)_{n-m}$
Hahn: $h_m^{(\alpha,\beta)}(x;N)$	$\binom{n}{m}\frac{(N-n)_{n-m}(n+\beta)^{[n-m]}}{(n+m+\alpha+\beta+1)^{[n-m]}}$
Hahn–Eberlein: $\tilde{h}_m^{(\mu,\nu)}(x;N)$	$\binom{n}{m}\frac{(N-n)_{n-m}(N-n+\nu)_{n-m}}{(\mu+\nu+2N-n-m-1)_{n-m}}$

Considering monic polynomials and the explicit expressions of these relations (5), (6) given in [3, 8], a solvable three term recurrence relation for the coefficients $I_m(n)$ is obtained for each classical discrete family. The corresponding solutions are listed in table 2.

Finally, it should be noted that the above requirements (a) and (b) for the P_n and Q_m families in (4) are expressed in terms of the difference operator (discrete case). However, the algorithm can also be applied if the difference operator is replaced by the derivative operator, and also if more general linear operators are considered (including Hahn operator). To illustrate this fact, let us consider the inversion problem (1) for some of the classical continuous orthogonal polynomials. In this continuous case, the most natural basis is x^n which satisfies the following first-order differential equation:

$$\mathcal{L}_1[x^n] \equiv x\,D(x^n) - n\,x^n = 0 \qquad \left(D \equiv \frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Since classical continuous orthogonal polynomials also satisfy relations of the type (5), (6), our algorithm works, giving recurrence relations for the coefficients $\widetilde{a}_{n,m}$ in (1). For the sake of completeness, we consider here the Bessel and Jacobi polynomials, the Laguerre and Hermite ones [1, 7] being well known.

1. Bessel polynomials: $Y_n^{(\alpha)}(x)$. Considering monic polynomials and the data of [5], the following recurrence relation for the coefficients $\widetilde{a}_{n,m}$ is obtained:

$$\begin{aligned}(\alpha+2m)(1+\alpha+2m)(2+\alpha+2m)^2(3+\alpha+2m)(-1+m-n)\widetilde{a}_{n,m-1}\\ -2m(1+\alpha+2m)(2+\alpha+2m)(3+\alpha+2m)(2+\alpha+2n)\widetilde{a}_{n,m}\\ -4m(1+m)(\alpha+2m)(2+\alpha+m+n)\widetilde{a}_{n,m+1} = 0\end{aligned}$$

valid for $1 \leqslant m \leqslant n$, with initial conditions $\widetilde{a}_{n,n+1} = 0$, $\widetilde{a}_{n,n} = 1$. This recurrence can be solved, giving

$$\widetilde{a}_{n,m} = (-1)^{n-m}\binom{n}{m}\frac{2^{n-m}}{(n+m+\alpha+1)^{[n-m]}} \qquad (0 \leqslant m \leqslant n).$$

Up to a change of normalization (monic polynomials have been considered here) this expression coincides with that given in [4], where only the case $\alpha = 0$ is studied. However, in this particular case, a misprint has been found in [6, p 73]. For any value of the Bessel parameter ($\alpha \neq -n, n \geqslant 2$), the expressions for these coefficients $\widetilde{a}_{n,m}$ have already been computed in [2, p 538, equation (7.5)], where a misprint is also present.

2. Jacobi polynomials: $P_n^{(\alpha,\beta)}(x)$. Again for monic polynomials, the recurrence given by our algorithm for the coefficients $\widetilde{a}_{n,m}$ in (1) in this case is

$$
\begin{aligned}
4m(1+m)(1+\alpha+m)(1+\beta+m)(\alpha+\beta+2m)(2+\alpha+\beta+m+n)\widetilde{a}_{n,m+1}& \\
+(\beta-\alpha)m(1+\alpha+\beta+2m)_3(2+\alpha+\beta+2n)\widetilde{a}_{n,m}& \\
+(\alpha+\beta+2m)_4(\alpha+\beta+2m+2)(m-n-1)\widetilde{a}_{n,m-1}&=0
\end{aligned}
$$

valid for $1 \leqslant m \leqslant n$, with initial conditions $\widetilde{a}_{n,n+1} = 0$, $\widetilde{a}_{n,n} = 1$. Solutions for some particular cases are given in table 3. Explicit hypergeometric representations for all (α, β) have been computed in [7, vol 1, p 277].

Table 3. Solutions of the inverse problem (1) for monic Chebyshev families.

(α,β)	$\widetilde{a}_{n,m}$ $(0 \leqslant m \leqslant n)$
$(\frac{1}{2},\frac{1}{2})$	$\begin{cases} 0 & \text{if } n-m=2k+1 \\ \binom{n}{k}\dfrac{m+1}{(n-k+1)4^k} & \text{if } n-m=2k \end{cases}$
$(-\frac{1}{2},-\frac{1}{2})$	$\begin{cases} 0 & \text{if } n-m=2k+1 \\ \binom{n}{k}2^{m-n+1} & \text{if } n-m=2k \end{cases}$
$(\frac{1}{2},-\frac{1}{2})$	$\left(-\frac{1}{2}\right)^{n-m}\binom{n}{[(n-m)/2]}$
$(-\frac{1}{2},\frac{1}{2})$	$\left(\frac{1}{2}\right)^{n-m}\binom{n}{[(n-m)/2]}$

This work was partially supported by NATO grant CRG-960213. AR also thanks the Universidad de Vigo for their kind invitation which allowed us to finish this letter.

References

[1] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[2] Al-Salam W A 1957 The Bessel polynomials *Duke Math. J.* **24** 529
[3] Area I, Godoy E, Ronveaux A and Zarzo A 1996 Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: discrete case *J. Comput. Appl. Math.* submitted
[4] Dickinson D 1954 On Lommel and Bessel polynomials *Proc. Am. Math Soc.* **5** 946
[5] Godoy E, Ronveaux, A Zarzo A and Area I 1996 Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case *J. Comput. Appl. Math.* submitted
[6] Grosswald E 1978 *Bessel Polynomials (Lecture Notes in Mathematics 698)* (Berlin: Springer)
[7] Luke Y L 1969 *The Special Functions and Their Approximations* 2 vols (New York: Academic)
[8] Nikiforov A F, Suslov S K and Uvarov V B 1991 *Classical Orthogonal Polynomials of a Discrete Variable* (Berlin: Springer)
[9] Godsil C D 1993 *Algebraic Combinatorics* (New York: Chapman and Hall)
[10] Ronveaux A, Belmehdi S, Godoy E and Zarzo A 1996 Recurrence relation approach for connection coefficients. Applications to classical discrete orthogonal polynomials *CRM Proceedings and Lecture Note Series* vol 9 (Providence, RI: American Mathematical Society) p 321

[11] Ronveaux A, Zarzo A and Godoy E 1995 Recurrence relations for connection coefficients between two families of orthogonal polynomials *J. Comp. Appl. Math.* **62** 67

[12] Smirnov Yu F 1996 Orthogonal polynomials of a discrete variable and quantum algebras $SU_q(2)$ and $SU_q(1,1)$. Hidden $sl(2)$ algebra of the finite difference equations *Proc. Int. Workshop on Orthogonal Polynomials in Mathematical Physics (Univ. Carlos III, Leganés, Madrid, Spain, 1996)* ed F Marcellán (Leganés: University of Carlos III) to appear

[13] Wolfram S 1993 *Mathematica Version 2.2* (Reading, MA: Addison-Welsley)